Automatically Improving Accuracy for Floating Point Expressions

Pavel Panchekha
Alex Sanchez-Stern
James Wilcox
Zach Tatlock

PLSE

University of Washington
Floating Point’s Wild Success
Floating Point’s Wild Success
Floating Point’s Wild Success
Floating Point’s Wild Success

Often floating point is close to real arithmetic
Floating Point’s Wild Success

Often floating point is close to real arithmetic

But not always!
Floating Point’s Wild Success

But not always!

Numerous articles retracted [Altman ’99, ’03]
Financial regulations [Euro ’98]
Market distortions [McCullough ’99, Quinn ’83]
Rounding Error in Sculpture
Rounding Error in Sculpture

Blake Courter
@bcourter
Rounding Error in Sculpture

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Rounding error
Rounding Error in Sculpture

Blake Courter
@bcourter
Numerical imprecision in complex square root #208

Merged josdejong merged 2 commits into josdejong:develop from pavpanchekha:develop on Aug 12, 2014

Conversation 1  Commits 2  Files changed 2

pavpanchekha commented on Aug 11, 2014

The expression used for complex square root returns imprecise results for negative reals. To avoid this imprecision, the equation is rearranged not to add \( r \) to \( x + re \) (which are of similar size and opposite sign).
Existing options
Existing options

- Unreliable
+ Fast Code
Existing options

- Unreliable
+ Fast Code
Existing options

- Unreliable
+ Fast Code

+ More Reliable
- Slow Code

![Diagram with characters and speech bubbles: "Hey! Get back to work!" and "Oh. Carry on." with a symbol "MPFR" and "Big Float".]
Existing options

- Unreliable
+ Fast Code

+ More Reliable
- Slow Code

+ Reliable
+ Fast Code
Existing options

- Unreliable
  + Fast Code
- Slow Code
+ More Reliable

+ Reliable
+ Fast Code
- Expert Task
Heuristic search to find expert transformations
Heuristic search to find expert transformations

Worked Example
How Herbie Works
Evaluation
Heuristic search to find expert transformations

Worked Example

How Herbie Works

Evaluation
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \quad \frac{2a}{2a}\]
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\]
Rounding Error in Quadratic

\[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

What is rounding error?
Rounding Error in Quadratic

\[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

What is rounding error?

exact

\[ [e]_R \]
What is rounding error?

exact

\[ [e]_R \]

computed

\[ [e]_F \]
Rounding Error in Quadratic

\[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

What is rounding error?

exact \[ [e]_R \]  computed \[ [e]_F \]

7 ULPs
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\]

What is rounding error?

exact \[ e_R \]  computed \[ e_F \]

7 ULPs

\[ \log(ULPs) \] estimates # of incorrect bits
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac}\]
\[\frac{2a}{2a}\]

\[\log(\text{ULPs})\]
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\]
Rounding Error in Quadratic

\[-b + \frac{\sqrt{b^2 - 4ac}}{2a}\]
Rounding Error in Quadratic

\[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]
Rounding Error in Quadratic

\[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\]  \[\Rightarrow\]

Overflow

If \( b \) is large, \([b^2]_F\) overflows and the whole expression returns \( \infty \).
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\]  

\[
\begin{cases} 
\frac{c}{b} - \frac{b}{a} & \text{if } b \in A 
\end{cases}
\]

Overflow

If \( b \) is large, \( \lceil b^2 \rceil_F \) overflows and the whole expression returns \( \infty \).
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\]

\[
\begin{align*}
\frac{c}{b} - \frac{b}{a} & \quad \text{if } b \in \text{A}
\end{align*}
\]
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\]

\[
\begin{cases} 
\frac{c}{b} - \frac{b}{a} & \text{if } b \in \mathbb{A} 
\end{cases}
\]

Pretty Accurate

\[
\text{log(Ulps)}
\]

\[
b
\]
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \quad \frac{c}{b} - \frac{b}{a} \quad \begin{cases} \frac{c}{b} - \frac{b}{a} & \text{if } b \in \mathbb{A} \\ \frac{-b + \sqrt{b^2 - 4ac}}{2a} & \text{if } b \in \mathbb{B} \end{cases} \]

Pretty Accurate

\[\log(\text{ULPs})\]
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\]

\[\begin{cases} \frac{c}{b} - \frac{b}{a} & \text{if } b \in \text{A} \\ \frac{-b + \sqrt{b^2 - 4ac}}{2a} & \text{if } b \in \text{B} \end{cases}\]
Rounding Error in Quadratic

\[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad \begin{cases} \frac{c}{b} - \frac{b}{a} & \text{if } b \in A \\ \frac{-b + \sqrt{b^2 - 4ac}}{2a} & \text{if } b \in B \end{cases} \]

Catastrophic Cancellation

If \( b \) is large, but \( a \) and \( c \) are small, \( b \approx \sqrt{b^2 - 4ac} \) and the difference is rounded off.
Rounding Error in Quadratic

\[
\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad \begin{cases} 
\frac{c}{b} - \frac{b}{a} & \text{if } b \in \textcolor{red}{A} \\
\frac{-b + \sqrt{b^2 - 4ac}}{2a} & \text{if } b \in \textcolor{blue}{B} \\
\frac{2c}{-b - \sqrt{b^2 - 4ac}} & \text{if } b \in \textcolor{blue}{C}
\end{cases}
\]

Catastrophic Cancellation

If \( b \) is large, but \( a \) and \( c \) are small, \( b \approx \sqrt{b^2 - 4ac} \) and the difference is rounded off.
Rounding Error in Quadratic

\[
\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad \begin{cases} 
\frac{c}{b} - \frac{b}{a} & \text{if } b \in A \\
\frac{-b + \sqrt{b^2 - 4ac}}{2a} & \text{if } b \in B \\
\frac{2c}{-b - \sqrt{b^2 - 4ac}} & \text{if } b \in C
\end{cases}
\]
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\] \quad \Rightarrow \quad \left\{ \begin{array}{l} \frac{c}{b} - \frac{b}{a} \\
\frac{-b + \sqrt{b^2 - 4ac}}{2a} \\
\frac{2c}{-b - \sqrt{b^2 - 4ac}} \end{array} \right. \quad \text{if } b \in A \\
\text{if } b \in B \\
\text{if } b \in C

Overflow again

\log(ULPs)
Rounding Error in Quadratic

\[-b + \frac{\sqrt{b^2 - 4ac}}{2a}\]

\[\frac{c}{b} - \frac{b}{a}\]

\[-b + \frac{\sqrt{b^2 - 4ac}}{2a}\]

\[-\frac{2c}{b}\]

\[-\frac{c}{b}\]

\[\text{Overflow again}\]

\[\text{if } b \in A\]

\[\text{if } b \in B\]

\[\text{if } b \in C\]

\[\text{if } b \in D\]
Rounding Error in Quadratic

\[ -b + \sqrt{b^2 - 4ac} \]
\[
\frac{2}{2a}
\]

\[ \begin{cases} 
\frac{c}{b} - \frac{b}{a} & \text{if } b \in A \\
\frac{-b + \sqrt{b^2 - 4ac}}{2a} & \text{if } b \in B \\
\frac{2c}{-b - \sqrt{b^2 - 4ac}} & \text{if } b \in C \\
-\frac{c}{b} & \text{if } b \in D 
\end{cases} \]
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\] 

\[\begin{cases} \frac{c}{b} - \frac{b}{a} & \text{if } b \in A \\ \frac{-b + \sqrt{b^2 - 4ac}}{2a} & \text{if } b \in B \\ \frac{2c}{-b - \sqrt{b^2 - 4ac}} & \text{if } b \in C \\ -\frac{c}{b} & \text{if } b \in D \end{cases}\]
Heuristic search to find expert transformations

**Worked Example**

How Herbie Works

Evaluation
Heuristic search to find expert transformations

Worked Example

How Herbie Works

Evaluation
Herbie Architecture

e sample candgens regimes e'

candgens

focus

generate more candidates
Herbie Architecture

- Sample
- Cands
- Focus
- Ground truth
- Regimes
- E'
- Generate more candidates

Diagram:
- E -> Cands
- Focus -> Cands
- Cands -> E'
- E sample -> Regimes -> E'
- Generate more candidates
Herbie Architecture

- Sample
- Cands
- Regimes
- Generate more candidates
- Focus
- Localize error
- \( e \)
- \( e' \)
Herbie Architecture

- **e** sample → **cands** regimes → **e’**
- **generate more candidates**
- **focus**
- **Heuristic search**

- **e**
- **cands**
- **e’**
Herbie Architecture

- Sample: e → cand
- Regimes: cand → e'
- Focus: e' → cand
- Generate more candidates: cand

Keep all good candidates: cand
Herbie Architecture

- **Sample**: $e$ → **Candidates**: $\text{cands}$
- **Regimes**: $\text{cands}$ → $e'$
- **Combine candidates**

- **Focus**: $e' \downarrow$
- **Generate more candidates**: $e' \uparrow$

$e'$ is the refined output after combining candidates through regimes.
Herbie Architecture

e \xrightarrow{sample} cands \xrightarrow{regimes} e'

Ground truth

focus

generate more candidates
Determine ground truth

\[ X = \text{sample}(\text{domain}(e)) \]

e.g. \[ X = \{1.2 \cdot 10^{-17}, -3.8 \cdot 10^{204}, 173.5, \ldots \} \]
Determine ground truth

\[ X = \text{sample}(\text{domain}(e)) \]

e.g. \( X = \{1.2 \cdot 10^{-17}, -3.8 \cdot 10^{204}, 173.5, \ldots \} \)

64 random bits
Determine ground truth

\[ X = \text{sample}(\text{domain}(e)) \]
e.g. \[ X = \{1.2 \cdot 10^{-17}, -3.8 \cdot 10^{204}, 173.5, \ldots \} \]

\[ \text{Round}([e]_\mathbb{R}(X)) \]

Get 64-bit prefix with MPFR.

**Subtle!** See paper.
Determine ground truth

\[ X = \text{sample}(\text{domain}(e)) \]

\[ \text{e.g. } X = \{1.2 \cdot 10^{-17}, -3.8 \cdot 10^{204}, 173.5, \ldots \} \]

\[ \text{Round} \left( \left\lfloor e \right\rfloor_{\mathbb{R}}(X) \right) \]

\[ \left\lfloor e \right\rfloor_{\mathbb{F}}(X) \]

Get 64-bit prefix with MPFR.

**Subtle!** See paper.

\[ \text{error} \]

\[ \text{e.g. } \{13.2b, 51.7b, 1b, \ldots \} \]

64 random bits

Compute in \(\mathbb{F}\)
Herbie Architecture

sample

cands

Ground truth

regimes

e'

focus

generate more candidates
Herbie Architecture

e \rightarrow \text{sample} \rightarrow \text{cands} \rightarrow \text{regimes} \rightarrow e' \leftarrow \text{focus} \leftarrow \text{generate more candidates} \\
\text{Localize error}
Focus: Estimate Error Source

\[-b + \sqrt{b^2 - 4ac} \over 2a\]
Focus: Estimate Error Source

1. For each op $f$ in $e$

\[
\frac{-b + \sqrt{b^2 - 4ac}}{2a}
\]
Focus: Estimate Error Source

1. For each op \( f \) in \( e \)
2. Evaluate args in \( \mathbb{R} \)

\[
\frac{-b + \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \begin{bmatrix} -b \end{bmatrix} \mathbb{R} \\
y = \begin{bmatrix} \sqrt{b^2 - 4ac} \end{bmatrix} \mathbb{R}
\]
1. For each op $f$ in $e$
2. Evaluate args in $\mathbb{R}$
3. Apply $f_{\mathbb{R}}$ to them

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$x = \begin{bmatrix} -b \end{bmatrix}_{\mathbb{R}}$

$y = \begin{bmatrix} \sqrt{b^2 - 4ac} \end{bmatrix}_{\mathbb{R}}$

$\text{Round}(x +_{\mathbb{R}} y)$
Focus: Estimate Error Source

1. For each op $f$ in $e$
2. Evaluate args in $\mathbb{R}$
3. Apply $f_\mathbb{R}$ to them
4. Apply $f_\mathbb{F}$ to them

\[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

\[
\begin{array}{c}
(x +_\mathbb{F} y) \quad \stackrel{+_\mathbb{F}}{\longrightarrow} \quad + \quad \stackrel{+_\mathbb{R}}{\longrightarrow} \quad \text{Round}(x +_\mathbb{R} y) \\
x = \begin{bmatrix} -b \end{bmatrix} \quad \mathbb{R} \quad y = \begin{bmatrix} \sqrt{b^2 - 4ac} \end{bmatrix} \quad \mathbb{R}
\end{array}
\]
Focus: Estimate Error Source

1. For each op \( f \) in \( e \)
2. Evaluate args in \( \mathbb{R} \)
3. Apply \( f_\mathbb{R} \) to them
4. Apply \( f_\mathbb{F} \) to them
5. Compare

\[
-b + \frac{\sqrt{b^2 - 4ac}}{2a}
\]

\[
(x +_\mathbb{F} y) \xrightarrow{+_\mathbb{F}} + \xrightarrow{+_\mathbb{R}} \text{Round}(x +_\mathbb{R} y)
\]

\[
x = \begin{bmatrix} -b \end{bmatrix} \mathbb{R}
\]

\[
y = \begin{bmatrix} \sqrt{b^2 - 4ac} \end{bmatrix} \mathbb{R}
\]
Focus: Estimate Error Source

1. For each op \( f \) in \( \mathbb{E} \)
2. Evaluate args in \( \mathbb{R} \)
3. Apply \( f_{\mathbb{R}} \) to them
4. Apply \( f_{\mathbb{F}} \) to them
5. Compare

\[
\begin{align*}
\frac{-b + \sqrt{b^2 - 4ac}}{2a}
\end{align*}
\]

\[
(x +_{\mathbb{F}} y) \quad +_{\mathbb{F}} \quad +_{\mathbb{R}} \quad \text{Round}(x +_{\mathbb{R}} y)
\]

\[
x = \begin{bmatrix}
-\frac{b}{2a}
\end{bmatrix} \quad \mathbb{R} \\
y = \begin{bmatrix}
\frac{\sqrt{b^2 - 4ac}}{2a}
\end{bmatrix} \quad \mathbb{R}
\]
Herbie Architecture

e sample → cands → regimes → e'  
focus  
Localize error  
generate more candidates
Herbie Architecture

e \rightarrow \text{cands} \rightarrow e' \rightarrow \text{e' through focus}

e \rightarrow \text{cands} \rightarrow \text{e' through generate more candidates}

Heuristic search
Herbie Architecture

Create candidates

- Sample
- Create candidates
- Focus
- Rewrite
- Series
- Simplify

\( e \rightarrow \text{cands} \rightarrow \text{regimes} \rightarrow e' \)
Apply rewrites to

\[
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
Apply rewrites to

\[
-\frac{b \pm \sqrt{b^2 - 4ac}}{2a}
\]

<table>
<thead>
<tr>
<th>Rule DB</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-x \quad \mapsto \quad 0 - x)</td>
</tr>
<tr>
<td>(x + y \quad \mapsto \quad \frac{x^2 - y^2}{x - y})</td>
</tr>
<tr>
<td>((x - y) + z \quad \mapsto \quad x - (y - z))</td>
</tr>
<tr>
<td>\ldots 120 more \ldots</td>
</tr>
</tbody>
</table>
Apply rewrites to \( b \pm \sqrt{b^2 - 4ac} \over 2a \)

<table>
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<td>((x - y) + z \leadsto x - (y - z))</td>
<td></td>
</tr>
</tbody>
</table>

... 120 more ...
Apply rewrites to

\[
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
-x \leadsto 0 - x
\]

\[
x + y \leadsto \frac{x^2 - y^2}{x - y}
\]

\[
(x - y) + z \leadsto x - (y - z)
\]

... 120 more ...

Rule DB
Apply rewrites to

\[- \frac{b \pm \sqrt{b^2 - 4ac}}{2a}\]

Recursive rewrites:
- Database of rules
- Flexible
- Chains of rewrites

Rule DB

- \(-x \leadsto 0 - x\)
- \(x + y \leadsto \frac{x^2 - y^2}{x - y}\)
- \((x - y) + z \leadsto x - (y - z)\)

… 120 more …
Apply rewrites to

\[-b \pm \sqrt{b^2 - 4ac}\]

\[\frac{-b}{2a}\]

Recursive rewrites:
- Database of rules
- Flexible
- Chains of rewrites

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<tr>
<td>… 120 more …</td>
</tr>
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</table>

No cancellation in denominator
Herbie Architecture

Create candidates

Sample

Cands

Rewrite

Series

Simplify

Focus

Regimes

E'}
Herbie Architecture

\[ e \xrightarrow{\text{sample}} \text{cands} \xrightarrow{\text{regimes}} e' \]

- focus
- rewrite
- series
- simplify

Approximate expr
Series Expansions

Idea: near-identities

\[-\frac{b + \sqrt{b^2 - 4ac}}{2a}\]
Series Expansions

Idea: near-identities

\[
\sqrt{1 - x} \approx 1 - \frac{x}{2}
\]
(for \( x \approx 0 \))
Series Expansions

Idea: *near*-identities

\[ \sqrt{1 - x} \approx 1 - \frac{x}{2} \]  
(for \( x \approx 0 \))

Bounded Laurent series:

- Transcendental functions
- Singularities
- Number of terms to take

\[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

\[ -b + b \left(1 - \frac{4ac}{2b^2}\right) \]
Herbie Architecture

e \rightarrow \text{sample} \rightarrow \text{cands} \rightarrow \text{regimes} \rightarrow e'

\downarrow \text{focus} \downarrow \text{rewrite} \downarrow \text{series} \downarrow \text{simplify}

\text{Approximate expr}
Herbie Architecture

e sample → cands → regimes → e' 

- focus
- rewrite
- series
- simplify

Cancel & clean up
Simplify Expressions

\[
\left( \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
\]
Simplify Expressions

\[
\left( \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
\]

\[
= \left( \frac{b^2 - (\sqrt{b^2 - 4ac})^2}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
\]

\[
= \left( \frac{b^2 - (b^2 - 4ac)}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
\]

\[
= \left( \frac{4ac}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
\]

\[
= \frac{2c}{-b - \sqrt{b^2 - 4ac}}
\]
Simplify Expressions

\[
\left( \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
\]

\[
= \left( \frac{b^2 - (\sqrt{b^2 - 4ac})^2}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
\]

\[
= \left( \frac{b^2 (b^2 - 4ac)}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
\]

\[
= \left( \frac{4ac}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
\]

\[
= \frac{2c}{-b - \sqrt{b^2 - 4ac}}
\]

Difficult! [Caviness '70]
- many possible rewrites
- huge search space
- avoid undoing progress!
Simplify Expressions

\[
\frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{-b - \sqrt{b^2 - 4ac}} / 2a
= \left( \frac{b^2 - (\sqrt{b^2 - 4ac})^2}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
= \left( \frac{b^2 - (b^2 - 4ac)}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
= \left( \frac{4ac}{-b - \sqrt{b^2 - 4ac}} \right) / 2a
= \frac{2c}{-b - \sqrt{b^2 - 4ac}}
\]

Difficult! [Caviness '70]
- many possible rewrites
- huge search space
- avoid undoing progress!

E-graphs [Nelson '79]
- Terminate early
- Prune useless nodes
- Restrict rewrites
Herbie Architecture

- Sample
- Cands
- Regimes
- E'
- Focus
- Rewrite
- Series
- Simplify

Keep all good candidates
Herbie Architecture

- Sample
- Cands
- Regimes
- Focus
- Rewrite
- Series
- Simplify

Combine candidates
Regime Inference

\[
\frac{c}{b} - \frac{b}{a} \quad \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \frac{2c}{-b - \sqrt{b^2 - 4ac}} \quad -\frac{c}{b}
\]
Regime Inference

\[ \frac{c}{b} - \frac{b}{a} \quad \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \frac{2c}{-b - \sqrt{b^2 - 4ac}} \quad -\frac{c}{b} \]
Regime Inference

\[
\left\{
\begin{align*}
\frac{c}{b} & - \frac{b}{a} \\
\frac{-b + \sqrt{b^2 - 4ac}}{2a} & \\
\frac{2c}{-b - \sqrt{b^2 - 4ac}} & \\
-\frac{c}{b}
\end{align*}
\right.
\]
Regime Inference

\[
\begin{aligned}
    \frac{c}{b} - \frac{b}{a} & \quad \text{if } b \in (\infty, -1.15E122] \\
    \frac{-b + \sqrt{b^2 - 4ac}}{2a} & \quad \text{if } b \in (-1.125E122, 1.06E-304] \\
    \frac{2c}{-b - \sqrt{b^2 - 4ac}} & \quad \text{if } b \in (1.06E-304, 4.62E63] \\
    -\frac{c}{b} & \quad \text{if } b \in (4.62E63, \infty)
\end{aligned}
\]

Dynamic programming:
- Bounds quickly
- Tune: binary search
- Pick best variable
Regime Inference

\[
\begin{align*}
\frac{c}{b} - \frac{b}{a} & \quad \text{if } b \in (\infty, -1.15\text{E}122] \\
\frac{-b + \sqrt{b^2 - 4ac}}{2a} & \quad \text{if } b \in (-1.125\text{E}122, 1.06\text{E}304] \\
\frac{2c}{-b - \sqrt{b^2 - 4ac}} & \quad \text{if } b \in (1.06\text{E}304, 4.62\text{E}63] \\
-\frac{c}{b} & \quad \text{if } b \in (4.62\text{E}63, \infty)
\end{align*}
\]

Dynamic programming:
- Bounds quickly
- Tune: binary search
- Pick best variable
Herbie Architecture

- Sample
- Combine candidates
- Focus
- Rewrite
- Series
- Simplify
Heuristic search to find expert transformations

Worked Example

How Herbie Works

Evaluation
Heuristic search to find expert transformations

Worked Example

How Herbie Works

Evaluation
Evaluating Herbie

A. Does accuracy improve?

B. Does it reproduce expert transformations?

C. Is the output code fast?

D. Does it work in the real world?
Examples from Hamming’s *NMSE*

**Chapter 3**: Function evaluation

28 worked examples & problems

- Quadratic formula (4)
- Algebraic rearrangement (12)
- Series expansion (12)
- Branches and regimes (2)
A. Improves accuracy in every test

Average bits correct (longer is better)
A. Improves accuracy in every test

Average bits correct (longer is better)

Accuracy of input

Accuracy of output
A. Improves accuracy in every test

- Accuracy of input
- Accuracy of output

Average bits correct (longer is better)

Dramatic improvement
B. Reproduces expert changes

Average bits correct (longer is better)
B. Reproduces expert changes

Of 12 with answers:
Same in 8
Different in 4

Average bits correct (longer is better)
B. Reproduces expert changes

- More accurate series expansion
- Handle overflow

Of 12 with answers:
  - Same in 8
  - Different in 4

Average bits correct (longer is better)
B. Reproduces expert changes

- More accurate series expansion
- No trig factorization
- Handle overflow
- Of 12 with answers:
  - Same in 8
  - Different in 4

Average bits correct (longer is better)
C. Output code is fast

Overhead CDF
(left is better)
D. Two MathJS Patches Accepted
D. Two MathJS Patches Accepted

**Numerical imprecision in complex square root #208**

Josdejong merged 2 commits into josdejong:develop from pavpanchekha:develop on Aug 12, 2014

Pavpanchekha commented on Aug 11, 2014

**Accuracy of sinh and complex cos/sin #247**

Josdejong merged 3 commits into josdejong:develop from pavpanchekha:complex-trig-accuracy on Dec 14, 2014

Pavpanchekha commented on Dec 12, 2014

The sin and cos function for complex arguments, and the sinh function for real arguments, are inaccurate when the inputs are very small. This is because $\text{Math.exp}(x) - \text{Math.exp}(-x)$ returns zero for small $x$, instead of the more accurate $2x$.

This patch replaces sinh by a Taylor expansion when the input is small, which increases accuracy.
D. Machine Learning Anecdote

I wasn't sure how to best rewrite [my] equations. **Herbie found numerically stable versions of the formulas**, and fixed all the divide-by-zero errors.

Clustering (bigger, darker blocks better)
Heuristic search to find expert transformations

Worked Example

How Herbie Works

Evaluation
Improve accuracy of floating point programs

Sampling to estimate error
Reduce global error to per-operation error
Iterative rewriting highest-error operations
Different expressions for different inputs

http://herbie.uwplse.org/
WELCOME TO THE SECRET ROBOT INTERNET

EARLIER...

Prove you are human:

0.1 + 0.2 = ?

0.300000000000000004
Herbie and Maximum Error

Often improved by Herbie:
Improvements large (28b) and small (.5b)
1+b improvement for 10/28 programs

Fewer high-error pts, same max error.

Bits error (histogram)
<table>
<thead>
<tr>
<th>Tool</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>FPDebug</td>
<td>Find inaccurate expressions</td>
</tr>
<tr>
<td>Herbie</td>
<td>Improve accuracy</td>
</tr>
<tr>
<td>Rosa</td>
<td>Prove accuracy satisfactory</td>
</tr>
<tr>
<td>FPTaylor</td>
<td></td>
</tr>
<tr>
<td>STOKE-FP</td>
<td>Optimize code</td>
</tr>
</tbody>
</table>
Error graphs along $a$ and $c$
Finding the rewrite rules

Standard mathematical identities:
  Commutativity, inverses, fractions, trig identities

No numerical methods knowledge

Don’t need to be true identities
  False rules do not improve accuracy
  Herbie will ignore them
Regimes often gains $\sim 15$ bits

Improvement from regimes (longer is better)

Dot : input program average accuracy
Bar : Herbie result w/out regimes